

B.Sc. Part I, Paper - II

Leibnitz's Theorem (Some Example)

1) If $y = (\sin^{-1}x)^2$, prove that

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0$$

Proof. we have $y = (\sin^{-1}x)^2$

differentiating both sides w.r.t. x , we get

$$y_1 = 2(\sin^{-1}x) \left(\frac{1}{\sqrt{1-x^2}} \right)$$

$$\Rightarrow y_1 \sqrt{1-x^2} = 2 \sin^{-1}x$$

$$\Rightarrow y_1^2 (1-x^2) = 4 (\sin^{-1}x)^2$$

$$\Rightarrow y_1^2 (1-x^2) = 4y \quad [\because y = (\sin^{-1}x)^2]$$

differentiating both sides w.r.t. x , we get

$$2y_1 y_2 (1-x^2) + y_1^2 (-2x) = 4y_1$$

$$\Rightarrow \cancel{2y_1} [y_2 (1-x^2) - x y_1] = \cancel{4y_1} \cdot 2$$

$$\Rightarrow y_2 (1-x^2) - x y_1 = 2$$

differentiating both sides n times w.r.t. x and using Leibnitz's theorem, we have

$$[y_{n+2} (1-x^2) + n C_1 y_{n+1} (-2x) + n C_2 y_n (2)] - [y_{n+1} x + n C_1 y_n (1)] = 0$$

$$\Rightarrow [y_{n+2} (1-x^2) + n y_{n+1} (-2x) + \frac{n(n-1)}{2} y_n (-2)] - [y_{n+1} x + n y_n] = 0$$

$$\Rightarrow y_{n+2} (1-x^2) - (2n+1)x y_{n+1} - [n(n-1) + n] y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0 \quad ; \text{ proved.}$$

which is reqd. result.

2) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2 - 1)y_{n+2} + (2n+1)x y_{n+1} + (n^2 - m^2)y_n = 0.$$

Proof:- Given, $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$

Diff. both sides w.r.t. x , we get

$$\frac{1}{m} y^{\frac{1}{m}-1} y_1 + \left(-\frac{1}{m}\right) y^{-\frac{1}{m}-1} y_1 = 2$$

$$\Rightarrow \frac{1}{m} \frac{y^{\frac{1}{m}}}{y} y_1 - \frac{1}{m} \frac{y^{-\frac{1}{m}}}{y} y_1 = 2$$

$$\Rightarrow y_1 [y^{\frac{1}{m}} - y^{-\frac{1}{m}}] = 2my$$

$$\Rightarrow y_1^2 [y^{\frac{1}{m}} - y^{-\frac{1}{m}}]^2 = 4m^2 y^2 \text{ [Squaring both sides]}$$

$$\Rightarrow y_1^2 [(y^{\frac{1}{m}} + y^{-\frac{1}{m}})^2 - 4y^{\frac{1}{m}} \cdot y^{-\frac{1}{m}}] = 4m^2 y^2$$

$$\Rightarrow y_1^2 [(2x)^2 - 4] = 4m^2 y^2$$

$$\Rightarrow y_1^2 (x^2 - 1) = m^2 y^2 \Rightarrow y_1^2 (x^2 - 1) = m^2 y^2$$

Diff. both sides w.r.t. x , we get

$$2y_1 y_2 (x^2 - 1) + y_1^2 (2x) = m^2 (2y y_1)$$

$$\Rightarrow 2y_1 [y_2 (x^2 - 1) + x y_1] = 2y_1 (m^2 y)$$

$$\Rightarrow y_2 (x^2 - 1) + x y_1 = m^2 y$$

Diff. n times on both sides w.r.t. x and by Leibnitz's theorem

$$[y_{n+2} (x^2 - 1) + n C_1 y_{n+1} (2x) + n C_2 y_n (2)] + [y_{n+1} x + n C_1 y_n (1)] = m^2 y_n$$

$$\Rightarrow [y_{n+2} (x^2 - 1) + n y_{n+1} (2x) + \frac{n(n-1)}{2} y_n (2)] + [y_{n+1} x + n y_n] = m^2 y_n$$

$$\Rightarrow y_{n+2} (x^2 - 1) + (2n+1) y_{n+1} + (n^2 - m^2) y_n = 0$$

∴ proved